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Composite wave interactions and the collapse of vacuums in gas dynamics

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ABSTRACT

We consider the p-system of isentropic gas dynamics. One of the outstanding questions in the study of one-dimensional Euler equations is the BV-existence and local structure of solutions having large data, including the vacuum state. The author has recently given a full description of pairwise wave interactions in 2×2 gas dynamics, which includes uniform interaction estimates up to vacuum. In this paper we consider composite interactions, which can be regarded as a degenerate superposition of pairwise interactions. We construct a class of weak solutions which demonstrate some interesting and surprising features, such as a shock of one family disappearing and a shock of the opposite family emerging. We give precise quantitative conditions which determine the outgoing waves. We also construct weak solutions of the p-system which demonstrate the collapse of a vacuum: in most cases two shocks will emerge from the vacuum, but in certain asymmetric cases a single shock and a rarefaction may emerge. We emphasize that the solutions constructed here are both explicit and exact weak solutions to the Euler equations of isentropic gas dynamics.

1. Introduction

We consider waves and their interactions in isentropic gas dynamics, in the large data regime. This means that nonlinear waves can have arbitrary wave strength, and solutions approach or even include the vacuum state, at which the equations become singular. Our goal in this paper is to demonstrate

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some interesting and surprising aspects of nonlinear interactions, and to understand the behavior of solutions when a vacuum collapses. We accomplish this by constructing a class of weak solutions, which contain strong nonlinear interactions and can include the vacuum state. We analyze these interactions exactly, and draw conclusions about the nature of interactions in general solutions. We emphasize that the solutions we construct are *exact, explicit* weak solutions of the isentropic Euler equations.

The author has recently given a complete description of pairwise Glimm interactions in the p-system [21]. In that work, because we consider waves of arbitrary strength, it is necessary to treat the states across waves exactly, while approximating the actual profiles of the waves: indeed, this is precisely what characterizes a Glimm interaction. In this paper we consider those interactions which can be realized exactly by carefully choosing an appropriate profile for the incident waves. Although some of them appear at first glance to be pairwise interactions, it is more appropriate to consider these as "composite" interactions, which occur when multiple pairwise interactions degenerate, so that the separate interactions take place at a single point of space–time.

An interesting example of one such interaction is that of a forward shock disappearing in finite time. The mechanism for this is as follows: if a backward compression wave collapses to form a backward shock (or if two backward shocks merge), a forward rarefaction is reflected as a result of shock formation. This forward reflected rarefaction can have arbitrary strength, provided the backward wave strength is large enough. Now, if we choose the profile of the backward compression wave in such a way that it collapses at a single point of space–time, and if the trajectory of a given forward shock includes that point, then the reflected forward wave is essentially the superposition of these two waves. Thus, if the backward wave is strong enough, the forward shock will disappear at the interaction point, while a backward shock emerges from that point. This disappearance of a shock is rather surprising, especially since, as is well known, in a scalar equation, once a shock forms it persists forever [6,2], either decaying as $1/t$ or being strengthened by interactions with nearby shocks. Moreover, we are able to give a precise condition which determines exactly the type and wave strengths of the outgoing waves.

We note that this effect can be repeated for weak waves, the main point being that when a compression collapses at a point, it has the potential to annihilate a shock at that point. It would be difficult to capture this effect without relying on an exact description of the waves, however: indeed, if the interactions were modeled asymptotically, as in Glimm's scheme or the usual front tracking methods [3,16,1,11], then truncation errors would persist and the wave would not disappear as in the exact solution. That said, our solutions have the structure identified by DiPerna [2], as the points of interaction are centers of simple waves.

This work is one part of the author's attempt to obtain global existence of large amplitude *BV* solutions to isentropic gas dynamics, including solutions containing vacuum states. At present, *BV* estimates are available only if the data $\|U_0\|_\infty$ is small enough [3,4], so that waves can be described asymptotically. Global *BV* estimates have yet to be proved for large amplitude data. In [21] and this paper, we obtain pairwise shock interaction estimates that are uniform up to the vacuum. These estimates need to be extended to take account of multiple wave interactions. Under regularity assumptions, in [9,17] it was shown that solutions do not approach vacuum in finite time, and these regularity assumptions would in turn be implied by *BV* bounds. To complete the proof of existence, both of these effects, namely strong shock interactions and approach to vacuum, need to be combined to produce the appropriate *BV* bound.

The rest of this paper addresses the collapse of a vacuum: in general, we cannot expect vacuums to persist for all time, and these will eventually collapse [17]. In physical space (Eulerian coordinates), the spatial extent of a "compressive" vacuum decreases until it vanishes. Our purpose is to describe the solution beyond this collapse of the vacuum. In general, two shocks should emerge from the point of vacuum collapse, but in some circumstances one of the emergent waves could be a rarefaction. Again, we obtain precise inequalities which determine when one of the outgoing waves is a rarefaction. As above, we construct exact data and solutions which include a decreasing vacuum region, and we continue this solution beyond the point of vacuum collapse. The key to our construction is the observation that although the wave adjacent to a vacuum is necessarily simple, it could be a compression. We thus set up the profiles of compressive waves so that the vacuum and two adjacent compressions all collapse at the same point. This is just the time-reversal of a Riemann problem containing a vacuum. The solution is then continued and analyzed by solving the resulting Riemann problem.

We consider the equations of isentropic gas dynamics in a Lagrangian frame,

$$\begin{pmatrix} v \\ u \end{pmatrix}_t + \begin{pmatrix} -u \\ p(v) \end{pmatrix}_x = 0, \tag{1}$$

where u is the fluid velocity, $v = 1/\rho$ is specific volume, p is the pressure. This system is strictly hyperbolic whenever $p'(v) < 0$, and the local sound speed $c = c(v)$ is defined by

$$c^2(v) = -p'(v). \tag{2}$$

We assume that the pressure is convex, $p''(v) > 0$, but our methods can be extended to nonconvex pressures with appropriate assumptions, see [15,18] and also [1,20]. The vacuum occurs at $\rho = 0$, or equivalently $v = \infty$, and we take $p(\infty) = c(\infty) = 0$. An equivalent condition for the system to admit the vacuum state is that the integral of the sound speed converges,

$$\int_1^\infty c(v)\,dv < \infty. \tag{3}$$

This is true for the most interesting physical systems, and we assume it throughout this paper. Our assumptions include the important case of a γ-law gas, which was analyzed in [21], but here we treat the general convex case as the extra scaling that holds for γ-law gases is not necessary for our construction. As in [21], we make the convenient nonlinear change of variables

$$h = h(v) \equiv \int_v^\infty c(v)\,dv, \tag{4}$$

which greatly simplifies our description of nonlinear waves. Moreover, in this variable, the vacuum corresponds to $h = 0$, which is easier to handle than $v \to \infty$. In fact, as in [21], we specify the equation of state by the wavespeed $c = c(h)$, which determines the pressure and specific volume by (7) below. We make one more technical assumption, namely that $c(h)$ be log-concave, that is

$$\frac{d^2}{dh^2} \log c(h) < 0 \quad \text{for } h > 0,$$

which we will see implies that a wave's strength increases when it crosses a shock.

It is well known that a vacuum can form in the solution to the Riemann problem [13]. Physically, the difference between the left and right velocities is so great that the gas cannot expand sufficiently, and a vacuum is formed. By analogy, a vacuum will form in a general solution if the initial data has a jump similar to that in which the Riemann solution contains a vacuum: under expected regularity assumptions, the author showed that this is the only way in which a vacuum can form in a solution [17]: that is, the initial velocity $u_0(x)$ is necessarily discontinuous at x_0, and the limiting Riemann data $(v_0, u_0)|_{x_0\pm}$ admits a vacuum in the solution of the corresponding Riemann problem.

The vacuum has previously been studied by several authors. In [9] and [8], the authors consider conditions under which the vacuum can and cannot form. In [10], Liu and Smoller considered the vacuum in an Eulerian frame of reference, and proved the existence of the Riemann problem in which one of the states corresponds to the vacuum, $\rho = 0$. In [12], a local existence theorem is obtained for the evolution of the boundary of the vacuum region. Wagner showed that the vacuum can equally be considered in a Lagrangian frame [14], where it corresponds to a δ-function in v. In contrast to [10], in [17] we introduced a modified "extended Riemann problem," in which the vacuum is embedded in between the states of a Riemann problem. This embedded vacuum appears as a δ-function in v, whose weight corresponds to the width of the vacuum in Eulerian coordinates (physical space). Physically, this allows us to treat vacuums of bounded spatial extent, and we can easily refer to the physical (spatial) width of the vacuum in Lagrangian coordinates. On the other hand, these extended Riemann problems do not always have a globally defined self-similar solution, see [19].

In [10] it is shown that a shock that is approaching a vacuum reaches the vacuum in finite time; in [17], we show that the shock is absorbed into the vacuum but changes the edge velocity; in particular, if the data is compactly supported, say, the vacuum will in general eventually become compressive, after which it will eventually collapse. Here we show by means of explicit examples what can happen when the vacuum collapses. We consider vacuums bounded by adjacent compression waves, and show that, provided these compressions are somewhat balanced, two shocks will emerge from the point of vacuum collapse. In particular, we can take a formal limit and treat the collapse of a vacuum as a formal interaction in itself. On the other hand, if one of the compressions is much stronger than the other, then the mechanisms described above come into play, and one of the emerging waves could be a rarefaction. Again we give an inequality in the incident wave parameters that determines the type and strength of the outgoing waves.

The paper is laid out as follows: in Section 2, we recall from [21] the global wave curve structure and definition of wave strengths. In Section 3, we recall the effects of pairwise Glimm interactions. In Section 4, we describe the profiles of the incident waves in our examples, and analyze several composite interactions. Finally, in Section 5, we describe the collapse of the vacuum and the consequences of this collapse. In Appendix A we prove monotonicity properties of wave curves for the general constitutive law, analogous to the results in [21] for a γ-law gas.

2. Wave structure

Making the change of variables (4) and using h as the thermodynamic variable, it is easy to see that

$$v'(h) = \frac{-1}{c(h)} \quad \text{and} \quad p'(h) = c(h). \tag{5}$$

As in [21] we make this change of variables explicit, by rewriting (1) as

$$\begin{pmatrix} v(h) \\ u \end{pmatrix}_t + \begin{pmatrix} -u \\ p(h) \end{pmatrix}_x = 0, \tag{6}$$

where the constitutive relation is determined by prescribing the Lagrangian sound speed $c(h)$. Using (5), we then describe the pressure and specific volume by

$$p(h) = \int_0^h c(\eta)\, d\eta \quad \text{and} \quad v(h) = \int^h \frac{-1}{c(\eta)}\, d\eta. \tag{7}$$

Note that v is given up to a constant, and the vacuum requirement (3) is divergence of the integral $\int d\eta/c$ as $h \to 0$. Moreover, calculus yields

$$\frac{dc}{dh} = \frac{dc}{dv}\frac{dv}{dh} = \frac{p''(v)}{2c^2} > 0, \tag{8}$$

so that convexity of $p(v)$ is monotonicity of $c(h)$.

Our change of coordinates is particularly simple for a γ-law gas,

$$p(v) = A_0 v^{-\gamma} \quad \text{with } \gamma > 1,$$

which corresponds to

$$c(h) = B_0 h^d, \tag{9}$$

where d is given by

$$d = \frac{\gamma+1}{\gamma-1}, \quad \text{and} \quad \gamma = \frac{d+1}{d-1}. \tag{10}$$

For a molecular gas, the ideal gas constant is given by $\gamma = 1 + 2/r$, where r is the number of degrees of freedom [7], and in this case (9) is a monomial with $d = r + 1$. The constitutive relation (9) yields a scaling symmetry as described in [21], but here we consider the more general convex case.

The advantage of using the variable h is that when we write the system in quasilinear form,

$$\begin{aligned} h_t + c(h)u_x &= 0, \\ u_t + c(h)h_x &= 0, \end{aligned} \tag{11}$$

it is symmetric hyperbolic, and readily yields the Riemann invariants $u \pm h$.

2.1. Simple waves

A simple wave is a (piece of a) solution that has a one-dimensional image: that is, it can be factored through a scalar parameter [5,6,13]. The states in a simple wave solve the equation

$$\frac{d}{d\epsilon}\begin{pmatrix} h \\ u \end{pmatrix} = r_\pm = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix},$$

where $r_\pm$ are the eigenvectors, and ϵ is the parameter, while the profile of the wave is determined by the scalar equation

$$\epsilon_t \pm c\big(h(\epsilon)\big)\epsilon_x = 0.$$

Taking $\epsilon = h$ and solving by characteristics, we see that across the wave the states satisfy

$$u - u_0 = \pm(h - h_0), \tag{12}$$

and constant states propagate along the forward (resp. backward) characteristics

$$\frac{dx}{dt} = \pm c(h). \tag{13}$$

The simple wave is a rarefaction if it is expanding, so that the wavespeed is increasing across the wave from left to right. Thus a backward rarefaction with left state $(h_0 u_0)^t$ satisfies

$$-c(h_0) \leqslant -c(h),$$

which, since $c'(h) > 0$, gives $h_0 \geqslant h$. Similarly, a forward simple wave with left state $(h_0 u_0)^t$ is a rarefaction if $h_0 \leqslant h$. On the other hand, compressions satisfy the same equation (12), but the wavespeed *decreases* from left to right across the wave. Thus $h_0 \leqslant h$ for a backward compression, and $h_0 \geqslant h$ for a forward compression.

We get a convenient consistent description of all simple waves by labeling the left, right, ahead and behind states of a simple wave, so that $a = l$ and $b = r$ for a backward wave, and $a = r$ and $b = l$ for a forward wave. Doing so, we get the concise description

$$u_r - u_l = h_a - h_b \tag{14}$$

for *all* simple waves, where the wave is rarefactive if $h_a > h_b$ and compressive if $h_b > h_a$.

2.2. Shocks

It is well known that compressions cannot be sustained, and shocks will form in the solution. These are determined by the Rankine–Hugoniot and entropy conditions [6,13]. The Rankine–Hugoniot equations are

$$s\big[v(h)\big] = -[u] \quad \text{and} \quad s[u] = \big[p(h)\big] \tag{15}$$

where as usual, $[\cdot]$ denotes the jump in states across the shock, and s is the shock speed. Solving, we get

$$s = \pm\sigma \quad \text{and} \quad u - u_0 = \pm K(h_0, h),$$

where σ is the absolute shock speed,

$$\sigma = \sigma(h_0, h) \equiv \sqrt{\frac{p(h_0) - p(h)}{v(h) - v(h_0)}}, \tag{16}$$

and the symmetric function K is defined by

$$K(h_1, h_2) = \sqrt{\big(p(h_2) - p(h_1)\big)\big(v(h_1) - v(h_2)\big)}. \tag{17}$$

If we again label left, right, ahead and behind states, then Lax's entropy condition can be expressed by absolute wavespeeds as

$$c(h_b) > \sigma(h_a, h_b) > c(h_a), \tag{18}$$

which simply states that absolute characteristic speed is greater behind the shock. This in turn implies $h_b > h_a$, consistent with compression, and the states across both forward and backward shocks satisfy

$$u_r - u_l = -K(h_a, h_b). \tag{19}$$

2.3. Shock error

When solving the Riemann problem, we use centered waves, which are those emanating from a single discontinuity at the origin. These can be shocks, which have no width, or rarefactions, all of whose characteristics meet at the origin. We will also treat centered compressions, which focus at a single point.

Defining the function $G : \mathbf{R}^2 \to \mathbf{R}$ by

$$G(h_1, h_2) = \begin{cases} h_1 - h_2, & \text{for } h_1 \geqslant h_2, \\ -K(h_1, h_2), & \text{for } h_1 \leqslant h_2, \end{cases} \tag{20}$$

we can then describe all shocks and centered rarefactions by

$$u_r - u_l = G(h_a, h_b). \tag{21}$$

Using this concise description of the waves allows us to understand the Riemann problem and wave interactions through the functions G and K. For more details, we refer the interested reader to [21].

In studying waves and their interactions, we want to measure the difference between the shock and rarefaction curves. To this end, we introduce the *shock error* function Θ, defined by the identity

$$G(h_1, h_2) = h_1 - h_2 - 2\Theta(h_1, h_2), \tag{22}$$

so that $\Theta(h_1, h_2) = 0$ for $h_1 \geqslant h_2$, and

$$\Theta(h_1, h_2) = \big(K(h_1, h_2) + h_1 - h_2\big)/2 \quad \text{for } h_1 \leqslant h_2. \tag{23}$$

It is clear that Θ is supported on shocks, and is a measure of the nonlinearity and convexity of the function $K(h_1, h_2)$, which in turn describes the shock curves.

We use the shock error to give a natural definition of wave strength, namely, we define the strength of a centered wave by

$$\Gamma(h_a, h_b) = h_a - h_b - \Theta(h_a, h_b). \tag{24}$$

With this definition, we follow the usual convention that rarefactions have positive strength while shocks have negative strength. Also note that (21), (22) yield

$$2\Gamma(h_a, h_b) = (u_r - u_l) + (h_a - h_b),$$

so that the (absolute) wave strength is the average of the (absolute) change in the coordinates u and h across a wave. This is also the change in the appropriate Riemann invariant across the wave.

Since Θ is supported only on shocks, we extend the use of the shock error to centered compressions by using a *shock indicator* function, χ_{ab}, which is 1 when the wave is a shock and 0 otherwise. We then describe the states across a general wave as

$$u_r - u_l = h_a - h_b - 2\chi_{ab}\Theta(h_a, h_b), \tag{25}$$

while the corresponding wavespeeds satisfy the characteristic condition (13) if $\chi_{ab} = 0$, and the shock condition

$$dx/dt = \pm\sigma(h_a, h_b) \quad \text{if } \chi_{ab} = 1,$$

where σ is given by (16).

We now state a lemma which describes the geometry of the shock error function. As this was proved in [21] in the special case of a γ-law gas, we give the general proof in Appendix A.

Lemma 1. *The wave function G, shock error Θ, and wave strength Γ are C^2 functions, monotone non-increasing in one variable and non-decreasing in the other, and in particular*

$$\frac{\partial}{\partial h_1}\Theta(h_1, h_2) \leqslant 0 \quad \textit{and} \quad \frac{\partial}{\partial h_2}\Theta(h_1, h_2) \geqslant 0. \tag{26}$$

Whenever $h_1 \leqslant h_2 \leqslant h_3$, we have

$$0 \leqslant \Theta(h_1, h_2) + \Theta(h_2, h_3) \leqslant \Theta(h_1, h_3) \tag{27}$$

with analogous statements for G, K and Γ. Moreover, if $c(h)$ is log-concave, we also have the inequality

$$\frac{\partial}{\partial h_1}\Theta(h_1,h_2)+\frac{\partial}{\partial h_2}\Theta(h_1,h_2)\leqslant 0, \tag{28}$$

with equality if and only if $h_1 \geqslant h_2$, with similar inequalities for G and Γ.

These inequalities have interpretations in terms of wave interactions: (27) is the condition that a rarefaction is reflected when shocks merge, while (28) implies that a wave's strength *increases* after it crosses an opposite shock.

2.4. Wave strength as variable

It is convenient to use the wave strength as the variable when considering interactions. To set this up, recall that the wave strength is

$$\Upsilon \equiv \Gamma(h_a,h_b) = h_a - h_b - \Theta(h_a,h_b),$$

and we use this to implicitly define the behind state

$$h_b \equiv \Phi(h_a,\Upsilon) = h_a - \Upsilon - \Theta\big(h_a,\Phi(h,\Upsilon)\big). \tag{29}$$

Note that if we regard the ahead state h_a as fixed, then Γ and Φ are inverse functions,

$$\Gamma(h_a,h_b)=\Upsilon \quad \text{iff} \quad \Phi(h_a,\Upsilon)=h_b. \tag{30}$$

We now describe the shock error in terms of wave strength as

$$\Omega(h_a,\Upsilon)\equiv\Theta\big(h_a,\Phi(h_a,\Upsilon)\big)=\Theta(h_a,h_b), \tag{31}$$

where h_a is the ahead state and $h_b=\Phi(h_a,\Upsilon)$ is behind. Note as above that since Θ is supported only on shocks, we have

$$\Omega(h_a,\Upsilon)=0 \quad \text{and} \quad \Phi(h_a,\Upsilon)=h_a-\Upsilon \quad \text{for } \Upsilon\geqslant 0.$$

Combining (29) and (31), we see that Φ and Ω are related by the identity

$$\Phi(h_a,\Upsilon)+\Omega(h_a,\Upsilon)=h_a-\Upsilon. \tag{32}$$

The functions Φ and Ω are defined and continuous up to the vacuum: indeed, although $\Theta(h_a,h_b)\to\infty$ as $h_a\to 0$ with h_b fixed, for fixed $\Upsilon<0$, we have

$$\Phi(h_a,\Upsilon)\to 0 \quad \text{and} \quad \Omega(h_a,\Upsilon)\to-\Upsilon \quad \text{as } h_a\to 0. \tag{33}$$

This is the best way to measure the shock error, as a shock preserves its strength as it moves through an opposite simple wave, and in particular as it approaches vacuum.

Properties analogous to those of Lemma 1 can be stated for Φ and Ω by simple calculus. We prove the following lemma in Appendix A:

Lemma 2. *For log-concave wavespeed $c(h)$, the functions Φ and Ω are C^2 and monotone in each variable, with derivatives satisfying the bounds*

$$-1 \leqslant \Phi_{;\Upsilon}(h, \Upsilon) \leqslant 0, \qquad 1 \leqslant \Phi_{;h}(h, \Upsilon), \tag{34}$$

and

$$-1 \leqslant \Omega_{;\Upsilon}(h, \Upsilon) \leqslant 0, \qquad \Omega_{;h}(h, \Upsilon) \leqslant 0, \tag{35}$$

respectively.

This boundedness of derivatives in wave strength (up to vacuum) means that our changes of variable have moved the singularity $v = \infty$ at the vacuum onto the *derivatives* $\Phi_{;h}$ and $\Omega_{;h}$ at $h = 0$. The following corollary follows immediately from (25), (29) and (31).

Corollary 3. *Any wave can be described in terms of its strength Υ as*

$$\begin{aligned} u_r - u_l &= \Upsilon - \chi\,\Omega(h_a, \Upsilon) = \Upsilon\bigl(1 + \chi\, R(h_a, \Upsilon)\bigr), \\ h_a - h_b &= \Upsilon + \chi\,\Omega(h_a, \Upsilon) = \Upsilon\bigl(1 - \chi\, R(h_a, \Upsilon)\bigr), \end{aligned} \tag{36}$$

where the χ is a shock indicator and the function $R(h_a, \Upsilon)$ satisfies the uniform bound

$$0 \leqslant R(h_a, \Upsilon) \leqslant 1,$$

and moreover $R(h_a, \Upsilon) = 0$ for all $\Upsilon \geqslant 0$.

2.5. Riemann problem

We briefly recall the solution of the Riemann problem, which is treated in detail in [21,18,17]; see also [13]. Thus we are given constant states $(h_l u_l)^t$ and $(h_r u_r)^t$ and wish to resolve the intermediate state $(h_* u_*)^t$, which in turn determines the outgoing centered waves and their strengths.

According to (21), we resolve the left and right waves as

$$u_* - u_l = G(h_l, h_*) \quad \text{and} \quad u_r - u_* = G(h_r, h_*),$$

respectively, and eliminating u_* yields the equation

$$u_r - u_l = G(h_l, h_*) + G(h_r, h_*), \tag{37}$$

which we solve for h_*. The wave strengths are then found using (24), and the profile of the centered waves are given by the corresponding characteristic equation

$$x = \pm c(h)t \quad \text{or} \quad x = \pm\sigma(h_{rl}, h_*)t$$

for rarefactions and shocks, respectively.

The function $G(h_0, h)$ is monotone decreasing with range $(-\infty, h_0]$, so (37) is uniquely solved provided

$$u_r - u_l \leqslant h_l + h_r. \tag{38}$$

If (38) fails, then a vacuum appears in the solution of the Riemann problem. The vacuum corresponds to $h = 0$ or $v = \infty$, and is resolved by allowing the specific volume v to become a Radon measure:

in the solution of the Riemann problem, this is necessarily a δ-function supported on the t-axis $x = 0$ (since $c(h) > 0$ for $h > 0$). Thus we can write

$$\nu_S = w(t)\delta(x)$$

for the singular part of v, and we obtain the weight $w(t)$ by weakly solving the volume equation

$$v_t - u_x = 0.$$

This yields

$$\frac{dw}{dt} = u_+ - u_-, \quad \text{so that} \quad w = (u_+ - u_-)t, \tag{39}$$

where $u_\pm$ are the right and left limits of u as we approach vacuum,

$$u_\pm = \lim_{h\to 0\pm} u = \lim_{x\to 0\pm} u(x,t).$$

This limit is independent of $t > 0$ by self-similarity, which also confirms that we have the correct scaling for the δ-function,

$$\nu_S = (u_+ - u_-)t\delta(x) = (u_+ - u_-)\delta(x/t). \tag{40}$$

The weight $w = (u_+ - u_-)t$ has a physical interpretation: namely, it describes the *spatial extent* of the vacuum as measured in the Eulerian coordinates of physical space. As well as being a consistent weak solution [14], this is consistent with the Eulerian interpretation of u as the fluid velocity: if $u_\pm$ are the edge velocities of the vacuum, then w is its spatial width.

We summarize in the following classical lemma [13,21]:

Lemma 4. *Given constant left and right states $(v_l u_l)^t$ and $(v_r u_r)^t$, respectively, there is a unique self-similar solution $(v(x,t), u(x,t))^t$ to the Riemann problem. If condition* (38) *holds, there is a unique intermediate state $(v_* u_*)^t$ which is a C^2 function of the data. If* (38) *fails, then for each fixed $t > 0$, the velocity $u(x,t)$ is a bounded monotone increasing function, while $v(x,t)$ is a Radon measure whose singular part is the Dirac measure* (40)*.*

We recall the observation from [10] that by the entropy condition (18), the vacuum can never appear as the state *behind* a shock wave. Also, a shock that approaches the vacuum will meet it in finite time. In [17], the author showed that the shock is absorbed into the vacuum, and the result of this absorption is that the corresponding edge velocity $u_\pm$ is changed. Thus, as a result of several such interactions, the vacuum itself can change from *expansive*, $u_+ > u_-$ (as in the solution of the Riemann problem), to *compressive*, $u_- > u_+$. As rarefactions cannot interact with vacuums, once a vacuum becomes compressive it will stay that way, and thus will always collapse ($w = 0$) in finite time. One of the goals of this paper is to describe by means of explicit examples what happens when the vacuum collapses.

3. Glimm interactions

Here we briefly recall the effects of pairwise Glimm interactions, fully analyzed in [21]. These are based on our exact description of the change in states across a nonlinear wave (25), and on the properties of the shock error function as stated in Lemma 1. Recall that a Glimm interaction is obtained by resolving the states in the interaction, while ignoring the actual wave profiles. This is equivalent to considering the asymptotic limit of the wave interaction in the self-similar scaling limit. Pairwise Glimm interactions were completely analyzed in [21], so we just recall the strategy and consider two generic interactions here: namely two waves of opposite families crossing, and two waves of the same family merging.

First we consider the crossing of two opposite waves. Thus we consider three incoming states, labeled by w, s and e, which determine two incident waves of strength $\Gamma(h_s, h_w)$ and $\Gamma(h_s, h_e)$, and we resolve the outgoing middle state n, which determines the outgoing wave strengths $\Gamma(h_w, h_n)$ and $\Gamma(h_e, h_n)$. Using (25), we describe the four waves as

$$\begin{gathered} u_s - u_w = h_s - h_w - 2\chi_{sw}\Theta(h_s, h_w), \qquad u_e - u_s = h_s - h_e - 2\chi_{se}\Theta(h_s, h_e), \\ u_n - u_w = h_w - h_n - 2\chi_{wn}\Theta(h_w, h_n), \quad \text{and} \\ u_e - u_n = h_e - h_n - 2\chi_{en}\Theta(h_e, h_n), \end{gathered} \tag{41}$$

where χ is a shock indicator. The key to the interaction is linearity in velocity u: eliminating u yields a scalar equation, which simplifies to

$$h_n + \chi_{wn}\Theta(h_w, h_n) + \chi_{en}\Theta(h_e, h_n) = h_w + h_e - h_s + \chi_{sw}\Theta(h_s, h_w) + \chi_{se}\Theta(h_s, h_e), \tag{42}$$

where we are solving for h_n. For a simple Glimm interaction, we set

$$\chi_{se} = \chi_{wn} \quad \text{and} \quad \chi_{sw} = \chi_{en}, \tag{43}$$

which says that an emerging wave is a shock if and only if the corresponding incident wave is a shock. We regard the other case, where one or both waves change, say from compression to shock, as a composite interaction, which will be analyzed in the next section.

From (43) and (42), we can write

$$\begin{aligned} &h_n - h_w + \chi_{se}\Theta(h_w, h_n) + \chi_{sw}\big(\Theta(h_e, h_n) - \Theta(h_e, h_w)\big) \\ &\quad = h_e - h_s + \chi_{se}\Theta(h_s, h_e) + \chi_{sw}\big(\Theta(h_s, h_w) - \Theta(h_e, h_w)\big). \end{aligned}$$

Now, by (26) of Lemma 1, the sides of this equality have the sign of $h_n - h_w$ and $h_e - h_s$, respectively, so that these in turn have the same sign. Together with (43), this implies that the outgoing wave $\Gamma(h_w, h_n)$ has exactly the same type (rarefaction, compression or shock) as the corresponding incident wave $\Gamma(h_s, h_e)$. Similarly, $\Gamma(h_e, h_n)$ has the same type as $\Gamma(h_s, h_w)$, and no wave can change its type in a simple Glimm interaction.

Next, we re-express (42) in terms of wave strengths, as

$$\begin{gathered} \Gamma(h_e, h_n) - \Gamma(h_s, h_w) = \chi_{se}\big(\Theta(h_w, h_n) - \Theta(h_s, h_e)\big) \quad \text{or} \\ \Gamma(h_w, h_n) - \Gamma(h_s, h_e) = \chi_{sw}\big(\Theta(h_e, h_n) - \Theta(h_s, h_w)\big), \end{gathered} \tag{44}$$

where we have used the *extended wave strength*,

$$\Gamma(h_a, h_b) \equiv h_a - h_b - \chi_{ab}\Theta(h_a, h_b), \tag{45}$$

defined for compressions, rarefactions and shocks. Since $\chi\Theta$ is supported only on shocks, we conclude immediately the important observation that *a wave's strength is unchanged if it crosses an opposite simple wave*. On the other hand, if the opposite wave is a shock, then the wave's strength does change: for example, suppose a simple wave (say $\Gamma(h_s, h_e)$) crosses a shock (whose strength $\Upsilon < 0$ does *not* change): then (44) yields

$$\Gamma(h_w, h_n) - \Gamma(h_s, h_e) = \Theta(h_e, h_n) - \Theta(h_s, h_w) = \Omega(h_e, \Upsilon) - \Omega(h_s, \Upsilon), \tag{46}$$

where we have used (31). Now, $\Gamma(h_s, h_e) = h_s - h_e$, and using (35), we conclude that the outgoing wave $\Gamma(h_w, h_n)$ is *stronger* than the incident simple wave.

Next, we consider the interaction of two waves of the same family. Since simple waves propagate with characteristic speed, they do not interact, and so at least one of the incident waves must be a shock, and the outgoing wave in the same family is also necessarily a shock. As above, we label the three incoming states (using labels l, m and r), and resolve the middle outgoing state, labeled $*$. Again by (25), we have, for backward waves,

$$u_m - u_l = h_l - h_m - 2\chi_{lm}\Theta(h_l, h_m), \qquad u_r - u_m = h_m - h_r - 2\chi_{mr}\Theta(h_m, h_r),$$
$$u_r - u_* = h_l - h_* - 2\chi_{l*}\Theta(h_l, h_*), \quad \text{and}$$
$$u_* - u_l = h_r - h_* - 2\chi_{r*}\Theta(h_r, h_*), \tag{47}$$

where χ is a shock indicator, with χ_{l*} and one or both of χ_{lm} and χ_{mr} taking the value 1. Moreover, if one of the incident waves is a rarefaction, then the interaction necessarily has finite width, and the reflected wave should have finite width, that is $\chi_{r*} = 0$. Again eliminating u, (47) simplifies to

$$h_* + \Theta(h_l, h_*) + \chi_{r*}\Theta(h_r, h_*) = h_r + \chi_{lm}\Theta(h_l, h_m) + \chi_{mr}\Theta(h_m, h_r),$$

where we are solving for h_*. We rewrite this as

$$h_* - h_r + \chi_{r*}\Theta(h_r, h_*) + \Theta(h_l, h_*) - \Theta(h_l, h_r)$$
$$= \chi_{lm}\Theta(h_l, h_m) + \chi_{mr}\Theta(h_m, h_r) - \Theta(h_l, h_r), \tag{48}$$

and observe by (26) that this has the sign of $h_* - h_r$, so that the type of the reflected wave is given by the difference of the Θ's. There are two cases: first, if both incident waves are compressive, then we have $h_l < h_m < h_r$, and by (27), the RHS of (48) is negative for all values of χ. Thus $h_r > h_*$, so the reflected wave is a rarefaction, and $\Theta(h_r, h_*) = 0$. On the other hand, suppose one of the incident waves is compressive. In this case, we impose the additional condition that $\sigma(h_l, h_m) < \sigma(h_m, h_r)$, which states that the wave behind has faster (average) wavespeed, which is clearly necessary for the interaction to take place. By (68) below, σ is symmetric and monotone, so this implies $h_l < h_r$ for backward waves. Since one wave is a rarefaction and one is a shock, we have either $h_m < h_l$ and $\chi_{mr} = 1$, or $h_r < h_m$ and $\chi_{lm} = 1$. In both cases, (26) and (48) yield $h_* > h_r$, and so the reflected wave is a *compression*, where we took $\chi_{r*} = 0$ since the reflected wave has finite width.

Noting that in all cases we have $\chi_{r*}\Theta(h_r, h_*) = 0$ in (48), we use (24) to express the interaction in terms of wave strengths: the reflected wave has strength

$$\Gamma(h_r, h_*) = \Theta(h_l, h_*) - \chi_{lm}\Theta(h_l, h_m) - \chi_{mr}\Theta(h_m, h_r), \tag{49}$$

while after simplifying, the transmitted wave has strength

$$\Gamma(h_l, h_*) = \Gamma(h_r, h_m) + \Gamma(h_m, h_l), \tag{50}$$

so that the strengths simply add. We summarize the Glimm interactions in the following lemma, also proved in [21]:

Lemma 5. *When two waves of the same family merge, a shock of that family results and a simple opposite wave is reflected. Moreover, the incident wave strengths add linearly as in* (50)*, while the reflected wave has signed strength given by* (49)*. If both incident waves are compressive, the reflected wave is a rarefaction, while if one incident wave is a rarefaction, then the reflected wave is a compression.*

If a wave of one family crosses a simple wave of the opposite family, its strength is unchanged during and after the interaction. If it crosses a shock of the other family, it emerges stronger, and the difference in its wave strength is given exactly by (44)*. In particular, no wave may change type by crossing a wave of the opposite family.*

We note that the collapse of a single compression wave can be treated in exactly the same way, so that the transmitted wave is a shock of the same strength, and the reflected wave is given by (49) with both $\chi = 0$.

4. Composite interactions

Recall that the Glimm interactions are obtained by resolving the intermediate states in an interaction, while leaving the characteristics unresolved. They are thus only approximate solutions to the system (1), (6). Here we will consider various wave interactions whose characteristics can be resolved exactly, and which thus do produce exact weak solutions. These should be regarded as composite interactions because they generally involve a (nonlinear) superposition of Glimm interactions.

4.1. Wave profiles

Because we wish to resolve all characteristics in the solution exactly, we construct solutions that consist entirely of centered waves, and we set these up in such a way that the interactions take place at a single point of space–time. In order to do this, we restrict our attention to incident waves which are either shocks or centered rarefactions, and place these waves so that all interactions occur at this space–time point.

That we can do this is easily accomplished by the following lemma, which describes the profile of a single centered compression focusing at the point $(x, t) = (0, 1)$; this can be changed to any (x_*, t_*) by scaling and translation of the data.

Lemma 6. *Suppose we are given an arbitrary ahead state* $(h_a u_a)^t$ *with* $h_a > 0$*, and wave strength* $\Upsilon < 0$ *or behind state* $h_b > h_a$ *satisfying*

$$h_b = h_a - \Upsilon.$$

Then we can find Cauchy data which produces a weak solution consisting of a forward or backward compression wave which focuses at $(x, t) = (0, 1)$*. In particular, the solution at time* $t = 1$ *consists of Riemann data, that is, we have*

$$(hu)^t(x, 1) = \begin{cases} (h_l u_l)^t, & x < 0, \\ (h_r u_r)^t, & x > 0. \end{cases}$$

Proof. We describe a backward compression; forward compressions are treated similarly. Thus $(h_l u_l)^t = (h_a u_a)^t$ and $(h_r u_r)^t = (h_b u_b)^t$, and, according to (14), we have

$$u_b = u_a + h_a - h_b = u_a + \Upsilon.$$

We now interpolate the states across the compression wave: that is, for $h \in [h_a, h_b]$, set

$$u(h) = u_a + h_a - h.$$

Since the compression is a backward simple wave, the (constant) state $(hu(h))^t$ propagates along its backward characteristic,

$$\frac{dx}{dt} = -c(h), \quad \text{that is} \quad x - x_* = c(h)(t_* - t).$$

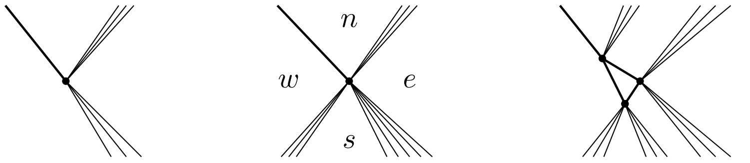

Fig. 1. Construction of Cauchy data.

We now insist that all of these characteristics go through the focus $(x_*, t_*) = (0, 1)$, and set $t = 0$; this simply yields $x(h) = c(h)$, which represents the position of the characteristic at $t = 0$. We can thus describe the full Cauchy data by

$$(h_0 u_0)^t(x) = \begin{cases} (h_l u_l)^t, & x < c(h_a), \\ (hu(h))^t, & c(h_a) \leqslant x = c(h) \leqslant c(h_b), \\ (h_r u_r)^t, & x > c(h_b), \end{cases}$$

where we can solve for $h(x)$ since $c'(h) = p > 0$. That the compression adjacent to constant states forms a weak solution follows classically [6]. The global weak solution is then obtained by solving the Riemann problem at $t = 1$. $\square$

We remark that by applying this lemma more than once, we can explicitly construct more general Cauchy data corresponding to multiple interactions as long as all the interactions in the global solution can be exactly resolved by Riemann problems, and all characteristics focus at isolated points. Examples of such characteristic diagrams are shown in Fig. 1. The first picture shows the case treated above, the second two compressions focusing at the same point, and the third shows a second compression focusing on the shock path, leading to more interactions. Note that the incoming and outgoing waves remain separated, as the edges of simple waves propagate with characteristic velocity, and so remain a fixed distance apart.

We also observe that it is easy to arrange the Cauchy data so that a given shock passes through the point $(x_*, t_*) = (0, 1)$, by taking the Cauchy data to have a single discontinuity at the appropriate point: thus, for a forward shock of speed $\sigma = \sigma(h_a, h_b)$, we place the discontinuity in the Cauchy data at $(x, t) = (-\sigma, 0)$.

We now consider the interaction shown in the middle of Fig. 1, where we assume that the Cauchy data has been set up using Lemma 6, so that all the characteristics and shock trajectories are exactly resolved. In order to apply the lemma, we require that $\Gamma(h_s, h_w)$ and $\Gamma(h_s, h_e)$ are both compressive, which is

$$h_e > h_s \quad \text{and} \quad h_w > h_s, \tag{51}$$

where these waves can be either shocks or compressions, and there are as yet no restrictions on the outgoing waves, as those are found from the resolution of the Riemann problem.

Since the incident waves have been chosen to focus at the single point $(x_*, t_*) = (0, 1)$, the resolution of the Riemann problem at that point leads to the global weak solution, and it simply remains to resolve the states and wave strengths in the interaction. However, this has already been done in our analysis of Glimm interactions: namely, the states satisfy Eq. (41) exactly, and the only change is that one or both of χ_{se} or χ_{sw} is taken to vanish, and, since the wave interaction takes place at a single point, the outgoing waves have zero width initially, so that $\chi_{wn} = \chi_{en} = 1$.

4.2. Interaction of two compressions

We thus proceed with our analysis by considering the various possible scenarios as the wave strengths vary. To begin, we suppose that both incident waves are compressions, so that $\chi_{se} = \chi_{sw} = 0$. This interaction can be regarded as a composite of three interactions, namely the separate focusing of the compressions into shocks and the crossing of the forward and backward waves. We now see that the focusing and crossing effects are in competition: namely, if a compressive wave crosses a shock wave, its strength increases, which is a generation of more compression, while if a compression focuses it generates opposite rarefaction. By construction, we are carefully choosing the data so that these effects are exactly superimposed, and we can study the results of this nonlinear superposition in detail.

In the present context, (42) becomes

$$h_n + \Theta(h_w, h_n) + \Theta(h_e, h_n) = h_w + h_e - h_s, \tag{52}$$

and recalling our definition of extended wave strength (45), this implies both

$$\Gamma(h_w, h_n) = \Gamma(h_s, h_e) + \Theta(h_e, h_n) \quad \text{and} \quad \Gamma(h_e, h_n) = \Gamma(h_s, h_w) + \Theta(h_w, h_n). \tag{53}$$

In particular, since $\Theta(h_a, h_b) \geqslant 0$, the outgoing wave is weaker than the corresponding incident wave: this is in contrast to Glimm interactions, and reflects the competition between interaction effects.

We first consider the symmetric interaction, in which $h_e = h_w$. Then (52) gives

$$h_w - h_n - 2\Theta(h_w, h_n) = h_s - h_e,$$

while (53) becomes

$$\Gamma(h_w, h_n) - \Theta(h_w, h_n) = \Gamma(h_s, h_e). \tag{54}$$

Now using (36), we write

$$\Gamma(h_w, h_n) - \Theta(h_w, h_n) = \Gamma(h_w, h_n)(1 + R),$$

where $R \in [0, 1]$; using this in (54) leads to

$$\Gamma(h_w, h_n) = \frac{\Gamma(h_s, h_e)}{1 + R} = \xi\,\Gamma(h_s, h_e), \tag{55}$$

for some unknown $1/2 \leqslant \xi \leqslant 1$. Thus in the symmetric case, the emerging shock is weaker than the incident compression by a factor ξ which is uniformly bounded.

We now assume without loss of generality that

$$h_e \geqslant h_w > h_s,$$

which by monotonicity of Γ is equivalent to

$$\Gamma(h_s, h_e) \leqslant \Gamma(h_s, h_w) < 0 \quad \text{and} \quad \Gamma(h_w, h_n) \leqslant \Gamma(h_e, h_n),$$

so the backward wave is stronger. Then $\Theta(h_e, h_w) = 0$ and (53) yields

$$\Gamma(h_w, h_n) + \Theta(h_e, h_w) - \Theta(h_e, h_n) = \Gamma(h_s, h_e) < 0,$$

so that monotonicity again implies that $h_n > h_w$, and we conclude that the outgoing backward wave is always a shock. On the other hand, we cannot draw the same conclusion for the outgoing forward wave: indeed, if we set $\Upsilon = \Gamma(h_w, h_n) < 0$, then using (31), we write (53) as

$$\Gamma(h_e, h_n) = \Gamma(h_s, h_w) + \Omega(h_w, \Upsilon).$$

Thus we conclude that if $\Upsilon < 0$ is large enough that

$$\Omega(h_w, \Upsilon) \geqslant -\Gamma(h_s, h_w) = h_w - h_s,$$

then $\Gamma(h_e, h_n) \geqslant 0$, $h_e \geqslant h_n$, and also (53) gives

$$h_s - h_e = \Gamma(h_s, h_e) = \Gamma(h_w, h_n) = \Upsilon.$$

Thus, if these conditions hold, the forward wave changes type across the interaction: that is, the incoming forward wave is a compression, while the outgoing wave is a rarefaction. Excluding translation by u, we obtain a three-parameter family of interactions, for which it is convenient to fix h_w and Υ.

Theorem 7. *Given fixed states $(h_w u_w)^t$ and strength $\Upsilon < 0$, let h_s satisfy*

$$\Phi(h_w, \Upsilon) + \Upsilon \leqslant h_s \leqslant h_w,$$

and set

$$h_n = \Phi(h_w, \Upsilon) \quad \textit{and} \quad h_e = h_s - \Upsilon.$$

Then the compression $\Gamma(h_s, h_w)$ changes into a rarefaction across the interaction of focusing compressions $\Gamma(h_s, h_w)$ and $\Gamma(h_s, h_e)$. In this case the wave strengths are

$$\Gamma(h_w, h_n) = \Gamma(h_s, h_e) = \Upsilon,$$
$$\Gamma(h_s, h_w) = h_s - h_w \quad \textit{and} \quad \Gamma(h_e, h_n) = h_s - \Upsilon - \Phi(h_w, \Upsilon).$$

Proof. First, since $\Phi(h_w, 0) = h_w$, (34) yields

$$\Phi(h_w, \Upsilon) \leqslant h_w - \Upsilon,$$

so that h_s can be chosen in an interval. Next, define h_n and h_e as above. Then the incoming compressions satisfy

$$\Gamma(h_s, h_e) = \Upsilon \quad \text{and} \quad \Gamma(h_s, h_w) = h_s - h_w,$$

and by (30), we have $\Gamma(h_w, h_n) = \Upsilon$. Moreover, by definition $h_e \geqslant h_n$, which yields

$$\Gamma(h_e, h_n) = h_e - h_n = h_s - \Upsilon - \Phi(h_w, \Upsilon).$$

It remains to show that the states as defined satisfy the interaction equation, which is any one of (52) or (53). Since $h_e \geqslant h_n$, $\Theta(h_e, h_n) = 0$ and so this becomes $\Gamma(h_w, h_n) = \Gamma(h_s, h_e)$, which holds by construction. □

The boundaries of the interval yield the extreme cases: first, $h_s = h_w$ reduces to the focusing of a single compression, which we do not regard as a composite interaction. Next, if $h_s = \Phi(h_w, \Upsilon) + \Upsilon$, then

$$h_e = h_s - \Upsilon = \Phi(h_w, \Upsilon) = h_n \quad \text{and} \quad \Gamma(h_e, h_n) = 0,$$

so there is no outgoing forward wave. This occurs when the incoming forward compression exactly balances the reflected rarefaction from the focusing backward compression. We note that this is an instance where the forward characteristics of a wave are terminated without entering a forward shock.

This is a large-data phenomenon and is in contrast to the structure of scalar equations and those generated by Glimm's method, in which the only way a characteristic terminates is by entering a corresponding shock wave. If h_s is outside this interval, $h_s < \Phi(h_w, \upsilon) + \Upsilon$, then both outgoing waves are shocks, with strength given by (53).

4.3. Shock–compression interaction

Next we consider the case in which the forward wave is a shock: this is analyzed in the same way, but we take $\chi_{sw} = 1$. In this case, (42) becomes

$$h_n + \Theta(h_e, h_n) + \Theta(h_w, h_n) = h_w + h_e - h_s + \Theta(h_s, h_w), \tag{56}$$

which yields

$$h_n - h_w + \Theta(h_w, h_n) + \Theta(h_e, h_n) - \Theta(h_e, h_w) = h_e - h_s + \Theta(h_s, h_w) - \Theta(h_e, h_w),$$

and again monotonicity of Θ implies $h_n > h_w$, so that the outgoing backward wave is a shock. As above, the reflected rarefaction from collapse of the backward wave competes with the forward shock, and again the outgoing forward wave depends on the relative strengths of these waves.

Theorem 8. *Given fixed states $(h_w u_w)^t$ and strength $\Upsilon < 0$, let h_s satisfy*

$$\Phi(h_w, \Upsilon) + \Upsilon \leqslant h_s - \Theta(h_s, h_w) \leqslant h_w, \tag{57}$$

and set

$$h_n = \Phi(h_w, \Upsilon) \quad \textit{and} \quad h_e = h_s - \Upsilon - \Theta(h_s, h_w).$$

Then the shock $\Gamma(h_s, h_w)$ changes into a rarefaction when it interacts exactly with the focusing compression $\Gamma(h_s, h_e)$.

Proof. The outgoing forward wave is a rarefaction provided the equivalent conditions

$$h_e \geqslant h_n, \qquad \Theta(h_e, h_n) = 0 \quad \text{and} \quad \Gamma(h_e, h_n) \geqslant 0 \tag{58}$$

hold, and (56) can then be written

$$h_e = h_s - \Theta(h_s, h_w) - \Gamma(h_w, h_n). \tag{59}$$

As before, we set $h_n = \Phi(h_w, \Upsilon)$, and we use (59) to define h_e. This ensures that the interaction is as stated, and it remains to check that h_e indeed satisfies (58), provided (57) holds; this again follows directly from (30) and the definitions of the states. $\square$

We remark that

$$\Gamma(h_w, h_n) = \Upsilon = h_s - h_e - \Theta(h_s, h_w) < h_s - h_e = \Gamma(h_s, h_e),$$

so that the outgoing shock is stronger than the incoming backward compression. Also, as above, the boundary case corresponds to $h_e = h_n$, so there is no outgoing forward wave. Here we again see that a wave terminates in finite time due to nonlinear superposition of interactions.

5. Collapse of the vacuum

We extend our analysis of composite interactions to a larger class of Cauchy data which includes the vacuum state. This demonstrates uniformity of the interaction effects described above and provides explicit examples that show exactly what happens when a vacuum collapses.

5.1. Profiles including vacuum

We begin by recalling that the vacuum is represented in Lagrangian variables as a δ-function in the velocity variable, whose weight $w(t)$ represents the spatial extent of the vacuum when expressed in Eulerian coordinates, while the thermodynamic variable h vanishes. Moreover, when the vacuum is bounded on both sides by noninteracting characteristics, then the weight $w(t)$ satisfies (39), namely

$$\frac{dw}{dt} = u_+ - u_-, \quad \text{where } u_\pm(t) = \lim_{x \to 0\pm} u(x,t),$$

and $u_\pm$ are piecewise constant because solutions are constant along characteristics.

We now set up Cauchy data that includes a vacuum of finite width, but which focuses to have zero width at $(x_*, t_*) = (0,1)$. If u_+ and u_- are given, and there are not interactions for $0 \leqslant t \leqslant 1$, then

$$w(t) = w_0 + (u_+ - u_-)t,$$

so we choose the initial weight to be

$$w_0 = u_- - u_+.$$

Here we require $u_- \geqslant u_+$, which means that in the language of [10], the vacuum is *compressive*: in particular, this vacuum cannot come from the solution of a Riemann problem. For consistency, we choose $\nu \equiv u_+ - u_- \leqslant 0$ as the parameter for the vacuum rather than $u_\pm$; this preserves the translation symmetry in u for these solutions. Next, we choose simple compressions which are adjacent to the vacuum, across which the thermodynamic variable continuously vanishes, while the velocity u satisfies (14). This is accomplished with a direct application of Lemma 6, where we take the limit of the ahead states, $h \to h_\pm = 0$, respectively.

Lemma 9. *Given arbitrary h_l, h_r and $\nu \leqslant 0$, there is Cauchy data which results in forward and backward compressions of strength $-h_l$ and $-h_r$, respectively, and which bracket a vacuum of initial weight $-\nu$, and all of which focus at the point $(x_*, t_*) = (0,1)$. Moreover, this Cauchy data can be translated by any constant velocity u_0.*

Proof. Picking an arbitrary u_l, we use Lemma 6 to generate Cauchy data for a forward compression from $(h_l u_l)^t$ to vacuum, $h_- = 0$, which yields

$$u_- = u_l + (h_- - h_l) = u_l - h_l,$$

and has strength $\Gamma(h_-, h_l) = -h_l$. Now, as described above, place a vacuum having (positive) weight $-\nu$ at the origin, which yields

$$u_+ = u_- + \nu = u_l - h_l + \nu,$$

and is compressive since $\nu \leqslant 0$. Finally, again use the lemma to place a backward compression adjacent to the vacuum with ahead state $(h_+ u_+)^t$ and ending at h_r, so that

$$u_r = u_+ + h_+ - h_r = u_l - h_l + \nu - h_r,$$

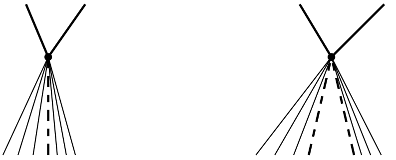

Fig. 2. Vacuum Cauchy data in Lagrangian and Eulerian frames.

with strength $\Gamma(h_+, h_r) = -h_r$. It follows from our construction that both waves and the vacuum focus at $(0,1)$. $\square$

It is instructive to describe the data in Eulerian coordinates: by a Galilean transformation, we can assume that the vacuum is placed symmetrically about the origin. Thus the left and right edges of the vacuum are at $y = \pm\nu/2$, respectively, where y is the spatial coordinate, and the left and right edge velocities are $u_\mp = \mp\nu$, respectively. The adjacent left compression lies in the wedge spanned by

$$y = (u_l + c_l)(t-1) \quad \text{and} \quad y = u_-(t-1),$$

and the right compression in the wedge given by

$$y = u_+(t-1) \quad \text{and} \quad y = (u_r - c_r)(t-1).$$

Here the lines are chosen to focus at $(y_*, t_*) = (1,0)$ and to satisfy the characteristic conditions for Eulerian coordinates, namely

$$\frac{dy}{dt} = u \mp c,$$

respectively. Fig. 2 shows one such profile in Lagrangian and Eulerian coordinates.

5.2. *Symmetric interaction*

We first consider the symmetric interaction, in which $h_l = h_r$. In particular, choosing $h_l = \epsilon$ small will give us an accurate picture of what happens in a general real solution when a vacuum collapses. In a general solution, we would expect the wave adjacent to the vacuum to be a rarefaction, even if the vacuum itself is compressive. To resolve the evolution after vacuum collapse in this more realistic case, we could approximate the actual vacuum by one with adjacent compressions of strength $-\epsilon$, and with rarefactions adjacent to those.

Suppose we are given $h_l = h_r = h_0$ and $\nu \leqslant 0$, and choose u to enforce maximal symmetry: according to the above, this means taking

$$u_+ = \nu/2 \quad \text{and} \quad u_- = -\nu/2,$$

and thus

$$u_l = -\nu/2 + h_0 \quad \text{and} \quad u_r = \nu/2 - h_0 = -u_l.$$

This gives $u_r - u_l = \nu - 2h_0$, and to resolve the outgoing waves we need to resolve the Riemann problem. Eqs. (37), (22) yield the equation

$$\nu - 2h_0 = 2h_0 - 2h_* - 4\Theta(h_0, h_*),$$

where we are solving for h_*. Expressed in terms of wave strengths, this becomes

$$\nu/2 + \Gamma(0, h_0) = \Gamma(h_0, h_*) - \Theta(h_0, h_*) = \Gamma(h_0, h_*)(1 + R), \tag{60}$$

where we have again used (36), so that

$$\Gamma(h_0, h_*) = \xi\bigl(\nu/2 + \Gamma(0, h_0)\bigr),$$

where the unknown ξ is uniformly bounded, $1/2 \leqslant \xi \leqslant 1$. This is the analogue of (55), with the extra term coming from the vacuum. In particular, since ν and $\Gamma(0, h_0)$ are both negative, we have the uniform bound

$$\Gamma(h_0, h_*) < \nu/4. \tag{61}$$

We conclude that when a compressive vacuum of weight $-\nu$ collapses, two shocks which have (unsigned) strength at least $|\nu/4|$ emerge from the point of vacuum collapse, independent of the size of the adjacent compressions.

Theorem 10. *When a compressive vacuum having weight $\nu < 0$ collapses, two shocks of strength $-\nu/4$ emerge symmetrically from the point of collapse.*

Proof. For $h_0 > 0$, write (60) as

$$\nu/2 + \Gamma(0, h_0) = \Gamma(h_0, h_*) - \Omega\bigl(h_0, \Gamma(h_0, h_*)\bigr).$$

Now, using (33), taking the limit $h_0 \to 0+$ yields

$$\nu/2 = 2 \lim \Gamma(h_0, h_*),$$

and the result follows. □

We note that, unlike our other examples, this limit cannot be realized exactly as a weak solution. In a weak solution containing the vacuum, the vacuum will be adjacent to a simple wave which is either a compression or rarefaction. In that case, we can regard the vacuum as instantaneously forming two shocks when the collapse occurs, and these shocks immediately interacting with the adjacent waves.

In particular, if a vacuum collapses while adjacent to rarefactions, then the shocks that emerge from the vacuum will immediately start to decay, which in turn reflects a compression, and this in turn will merge with the opposite shock, slowing down the decay of that wave. Moreover, since the wavespeeds approach 0 as $h_0 \to 0$, we expect the shock trajectories to form a cusp at (x_*, t_*).

5.3. Asymmetric interaction

We now consider the general case of the collapse of a vacuum of weight $\nu < 0$, adjacent to two compressions, all of which focus at $(x_*, t_*) = (0, 1)$. Thus the left compression, vacuum and right compression, respectively, satisfy

$$u_- - u_l = 0 - h_l, \qquad u_+ - u_- = \nu, \quad \text{and} \quad u_r - u_+ = 0 - h_r,$$

and we suppose that $h_l \geqslant h_r$, so the forward wave is stronger. Again eliminating u and resolving the Riemann problem at $t = 1$, we obtain

$$\nu - h_l - h_r = h_l - h_* - 2\Theta(h_l, h_*) + h_r - h_* - 2\Theta(h_r, h_*), \tag{62}$$

which is the asymmetric version of (60). We rewrite this as

$$h_* - h_r + \Theta(h_r, h_*) + \Theta(h_l, h_*) - \Theta(h_l, h_r) = h_l - \nu/2 > 0,$$

and conclude that the outgoing forward wave is always a shock. On the other hand, if h_l is large enough (relative to h_r and ν), then the reflected rarefaction from the focusing forward compression may overwhelm the backward shocks, and we can again get a single shock emerging from the vacuum.

Theorem 11. *If h_l, h_r and ν satisfy the relation*

$$\Theta(h_r, h_l) \geqslant h_r - \nu/2, \tag{63}$$

then the backward wave emerging from the point of collapse $(x_*, t_*) = (0, 1)$ is a rarefaction; otherwise it is a shock. If (63) holds, the wave strengths of the emerging waves are explicitly given by

$$\Gamma(h_r, h_*) = \nu/2 + \Gamma(h_-, h_l) < 0 \quad \textit{and}$$
$$\Gamma(h_l, h_*) = \nu/2 + \Gamma(h_+, h_r) + \Omega\big(h_r, \Gamma(h_r, h_*)\big),$$

respectively.

Proof. We rewrite (62) as

$$h_* - h_l + \Theta(h_l, h_*) + \Theta(h_r, h_*) - \Theta(h_r, h_l) = h_r - \nu/2 - \Theta(h_r, h_l),$$

and note that by monotonicity of Θ, $h_* \leqslant h_l$ if and only if (63) holds. Since $\Gamma(h_l, h_*)$ is centered, the first part follows. Next, if (63) holds, then $\Theta(h_l, h_*) = 0$, so (62) can be written

$$\Gamma(h_r, h_*) = h_r - h_* - \Theta(h_r, h_*) = \nu/2 - h_l,$$

or equivalently

$$\Gamma(h_l, h_*) = h_l - h_* = \nu/2 - h_r + \Theta(h_r, h_*).$$

The result now follows when we recognize that

$$\Gamma(h_-, h_l) = h_- - h_l = -h_l \quad \text{and} \quad \Gamma(h_+, h_r) = h_+ - h_r = -h_r$$

are the wave strengths of the incoming compressions. □

Appendix A. Properties of wave curves

Here we generalize results proved for a γ-law gas in [21] on the convexity of the wave curves, as stated in Lemma 1 above. In the general case, the wave functions do not scale and the results here are not as sharp as those obtained in [21]. However, the basic monotonicity properties of wave curves continue to hold. First we prove Lemma 1:

Lemma 12. *The wave function G, shock error Θ, and wave strength Γ are C^2 functions, monotone non-increasing in one variable and non-decreasing in the other. We have*

$$0 \leqslant \Theta(h_1, h_2) + \Theta(h_2, h_3) \leqslant \Theta(h_1, h_3) \tag{64}$$

whenever $h_1 \leqslant h_2 \leqslant h_3$, with analogous statements for G, K and Γ. Moreover, if $c(h)$ is log-concave, we also have the inequality

$$\Theta_{;1}(h_1, h_2) + \Theta_{;2}(h_1, h_2) \leqslant 0, \tag{65}$$

with equality if and only if $h_1 \geqslant h_2$, with similar inequalities for G and Γ.

Proof. It suffices to prove the lemma for one of the functions G, Θ and Γ. It is clear that these functions are smooth away from $h_2 = h_1$, and nonlinear only in the region $h_2 \geqslant h_1$. For smoothness, we thus need to calculate derivatives of $K(h_1, h_2)$ as $h_2 \to h_1+$. Using (17) and (16), we write

$$K(h_1, h_2)\sigma(h_1, h_2) = p(h_2) - p(h_1) \quad \text{and} \quad K(h_1, h_2)/\sigma(h_1, h_2) = v(h_1) - v(h_2), \tag{66}$$

and note that

$$K(h_1, h_2) \to 0, \qquad \sigma(h_1, h_2) \to c(h_1) \quad \text{as } h_2 \to h_1,$$

so that G is continuous by (20).

Now use (5) to calculate the derivatives

$$K_{;1} = \frac{-1}{2}\left(\frac{\sigma}{c(h_1)} + \frac{c(h_1)}{\sigma}\right) \quad \text{and} \quad K_{;2} = \frac{1}{2}\left(\frac{\sigma}{c(h_2)} + \frac{c(h_2)}{\sigma}\right), \tag{67}$$

so that

$$K_{;1}(h_1, h_2) \to -1 \quad \text{and} \quad K_{;2}(h_1, h_2) \to 1 \quad \text{as } h_2 \to h_1+,$$

which implies that G is C^1.

Similar calculations yield

$$\sigma_{;1} = \frac{\sigma}{2K}\left(\frac{\sigma}{c(h_1)} - \frac{c(h_1)}{\sigma}\right) \quad \text{and} \quad \sigma_{;2} = \frac{\sigma}{2K}\left(\frac{c(h_2)}{\sigma} - \frac{\sigma}{c(h_2)}\right) \tag{68}$$

and in particular, by Lax's entropy condition (18), these are both positive. We can also differentiate (16) in v, to get

$$2\sigma \frac{\partial}{\partial v_1}\sigma(v_1, v_2) = \frac{p'(v_1)(v_2 - v_1) + p(v_1) - p(v_2)}{(v_2 - v_1)^2} = \frac{-1}{2} p''(\nu)$$

for some ν, by Taylor's theorem. Thus, using (8),

$$\sigma_{;1}(h_1, h_2) = \frac{-1}{c(h_1)} \frac{\partial \sigma}{\partial v_1} = \frac{c^2(\eta)c'(\eta)}{2c(h_1)\sigma},$$

where $\eta = h(\nu)$, and we conclude the well-known fact

$$\sigma_{;1}(h_1, h_2) \to \frac{1}{2} c'(h_1) \quad \text{as } h_2 \to h_1.$$

Differentiating (67) again and using (68), we get

$$K_{;12} = \frac{-1}{2}\left(\frac{1}{c(h_1)} - \frac{c(h_1)}{\sigma^2}\right)\sigma_{;2} = \frac{-1}{4K}\left(\frac{\sigma}{c(h_1)} - \frac{c(h_1)}{\sigma}\right)\left(\frac{c(h_2)}{\sigma} - \frac{\sigma}{c(h_2)}\right), \tag{69}$$

while

$$K_{;11} = \frac{-1}{2}\left(1 - \frac{c^2(h_1)}{\sigma^2}\right)\left(\frac{\sigma}{c(h_1)}\right)_{;1} \quad \text{and} \quad K_{;22} = \frac{1}{2}\left(1 - \frac{c^2(h_2)}{\sigma^2}\right)\left(\frac{\sigma}{c(h_2)}\right)_{;2}.$$

It follows that all three second derivatives of K vanish as $h_2 \to h_1+$, which in turn implies that G is C^2. Since Θ and Γ differ from G by linear functions, these are also C^2.

Further, (23) and (67) yield

$$2\Theta_{;1} = \frac{-1}{2}\left(\frac{c(h_1)}{\sigma} + \frac{\sigma}{c(h_1)}\right) + 1 < 0 \quad \text{and} \quad 2\Theta_{;2} = \frac{1}{2}\left(\frac{c(h_2)}{\sigma} + \frac{\sigma}{c(h_2)}\right) - 1 > 0, \tag{70}$$

so that Θ is monotone decreasing in the first variable and increasing in the second. Since $\Theta(h_1, h_2) = 0$ for $h_1 \geqslant h_2$, this implies also that Θ is non-negative. It follows from Eqs. (22) and (24) that G and Γ are also monotone in each variable.

Next, suppose we are given $h_1 \leqslant h_2 \leqslant h_3$. Then we write

$$K(h_1, h_2) + K(h_2, h_3) - K(h_1, h_3) = \int_{h_1}^{h_2}\int_{h_2}^{h_3} K_{;12}(\zeta, \eta)\, d\eta\, d\zeta,$$

and, since $\zeta \leqslant \eta$, by (18) we have

$$c(\zeta) < \sigma(\zeta, \eta) < c(\eta),$$

so that by (69), $K_{;12}(\zeta, \eta) \leqslant 0$. Thus we conclude that

$$0 \leqslant K(h_1, h_2) + K(h_2, h_3) \leqslant K(h_1, h_3),$$

and the analogous inequalities hold for G and Θ, namely

$$\begin{aligned} 0 \geqslant G(h_1, h_2) + G(h_2, h_3) \geqslant G(h_1, h_3) \quad \text{and} \\ 0 \leqslant \Theta(h_1, h_2) + \Theta(h_2, h_3) \leqslant \Theta(h_1, h_3), \end{aligned} \tag{71}$$

since these differ from K by linear operations.

Finally, to show (65), we use (70) to write

$$\Theta_{;1} + \Theta_{;2} = \frac{1}{4} f\left(\frac{c(h_2)}{\sigma}\right) - \frac{1}{4} f\left(\frac{\sigma}{c(h_1)}\right),$$

where $f(y) = y + 1/y$. Now since $f(y)$ is monotone increasing for $y > 1$, the result (65) follows as long as we can show

$$\frac{\sigma}{c(h_1)} > \frac{c(h_2)}{\sigma} > 1. \tag{72}$$

The second inequality follows from (18), and the first is clearly equivalent to the condition $c(h_1)c(h_2) \leqslant \sigma(h_1, h_2)^2$, which is (77), proved in Lemma 14 below. $\square$

Before proving (72), we obtain Lemma 2 as a corollary:

Corollary 13. *For log-concave wavespeed $c(h)$, the functions Φ and Ω are C^2 and monotone in each variable, with derivatives satisfying the bounds*

$$-1 \leqslant \Phi_{;\Upsilon}(h,\Upsilon) \leqslant 0, \qquad 1 \leqslant \Phi_{;h}(h,\Upsilon), \tag{73}$$

and

$$-1 \leqslant \Omega_{;\Upsilon}(h,\Upsilon) \leqslant 0, \qquad \Omega_{;h}(h,\Upsilon) \leqslant 0, \tag{74}$$

respectively.

Proof. According to (29), Φ is given by

$$\Phi(h,\Upsilon) = h - \Upsilon - \Theta\big(h, \Phi(h,\Upsilon)\big).$$

Differentiating, we have

$$\Phi_{;\Upsilon}(h,\Upsilon) = -1 - \Theta_{;2}\Phi_{;\Upsilon}(h,\Upsilon) \quad \text{and} \quad \Phi_{;h}(h,\Upsilon) = 1 - \Theta_{;1} - \Theta_{;2}\Phi_{;h}(h,\Upsilon),$$

where the derivatives $\Theta_{;\cdot}$ are evaluated at (h,Φ). Solving, we get

$$\Phi_{;\Upsilon}(h,\Upsilon) = \frac{-1}{1+\Theta_{;2}} \quad \text{and} \quad \Phi_{;h}(h,\Upsilon) = \frac{1-\Theta_{;1}}{1+\Theta_{;2}}, \tag{75}$$

and inequalities (73) follow from (70), (65).

Next, (32) gives

$$\Omega(h,\Upsilon) = h - \Upsilon - \Phi(h,\Upsilon),$$

so that

$$\begin{aligned} \Omega_{;\Upsilon}(h,\Upsilon) &= -1 - \Phi_{;\Upsilon}(h,\Upsilon) = \frac{-\Theta_{;2}}{1+\Theta_{;2}} \quad \text{and} \\ \Omega_{;h}(h,\Upsilon) &= 1 - \Phi_{;h}(h,\Upsilon) = \frac{\Theta_{;1}+\Theta_{;2}}{1+\Theta_{;2}}, \end{aligned} \tag{76}$$

and (74) follows from (70), (65). $\square$

Finally, we prove (72), which is the condition that a wave's strength increases after it crosses a shock. Note that this is the only place we use the assumption that the wavespeed $c(h)$ is log-concave.

Lemma 14. *If the wavespeed $c(h)$ is log-concave, then*

$$c(h_1)c(h_2) \leqslant \sigma(h_1,h_2)^2, \tag{77}$$

for all $h_2 > h_1$.

Proof. Fix $h_1 < h_2$, and, using (5), write

$$p(h_2) - p(h_1) = \int_{h_1}^{h_2} c(h)\,dh \quad \text{and} \quad v(h_1) - v(h_2) = \int_{h_1}^{h_2} \frac{1}{c(h)}\,dh.$$

Thus, by (16),

$$\sigma(h_1, h_2)^2 = \frac{\int_{h_1}^{h_2} c(h)\,dh}{\int_{h_1}^{h_2} \frac{1}{c(h)}\,dh},$$

and our inequality (77) is thus equivalent to

$$I_A \equiv \int_{h_1}^{h_2} \frac{c(h)}{\sqrt{c(h_1)c(h_2)}}\,dh > \int_{h_1}^{h_2} \frac{\sqrt{c(h_1)c(h_2)}}{c(h)}\,dh \equiv I_B. \tag{78}$$

We now define the function

$$f(h) = \log \frac{c(h)}{\sqrt{c(h_1)c(h_2)}},$$

which is concave since $c(h)$ is log-concave. Then we have

$$I_A = \int_{h_1}^{h_2} e^{f(h)}\,dh \quad \text{and} \quad I_B = \int_{h_1}^{h_2} e^{-f(h)}\,dh,$$

and moreover,

$$f(h_1) = \log \sqrt{\frac{c(h_1)}{c(h_2)}} = -f(h_2).$$

Using the change of variables $k = h_1 + h_2 - h$, integral I_B becomes

$$I_B = \int_{h_2}^{h_1} e^{-f(h_1+h_2-k)}\,d(-k) = \int_{h_1}^{h_2} e^{g(k)}\,dk,$$

where we have set $g(h) \equiv -f(h_1 + h_2 - h)$, so that

$$g(h_1) = -f(h_2) = f(h_1) \quad \text{and} \quad g(h_2) = -f(h_1) = f(h_2),$$

and now g is convex.

Now, since f is concave, g is convex and these have the same values at the endpoints h_1 and h_2, it follows easily that

$$g(h) \leqslant f(h) \quad \text{for all } h \in [h_1, h_2],$$

and this in turn implies $I_B \leqslant I_A$, as required. □

These results were proved in [21] for the special case $c(h) = h^d$, $d > 1$, corresponding to a γ-law gas. It is also immediate from the proof that $c(h) = e^{\alpha h}$ is the limiting case for which equality holds in (77) (and thus also (65)); this corresponds to the isothermal case $\gamma = 1$ in which the vacuum does *not* appear.

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